Theory of Concepts and Deductive Logic

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What is a concept?

Voishvillo E.K. **Concept**. Moscow University Publishing, 1967.

Voishvillo E.K. **Concept as a Form of Thought**. Moscow University Publishing, 1989.

A *concept* is a construction of the type
$\alpha_1, \alpha_2, ..., \alpha_n A(\alpha_1, \alpha_2, ..., \alpha_n)$,

where
$\alpha_1, \alpha_2, ..., \alpha_n$ is a sequence of variables
and
$A(\alpha_1, \alpha_2, ..., \alpha_n)$ is a formula with free occurrences of $\alpha_1, \alpha_2, ..., \alpha_n$.

Intension and extension of a concept

Formula $A(\alpha_1, \alpha_2, ..., \alpha_n)$ expresses the *intension* of a concept $\alpha_1, \alpha_2, ..., \alpha_n A(\alpha_1, \alpha_2, ..., \alpha_n)$.

The *extension* of a concept $\alpha_1, \alpha_2, ..., \alpha_n A(\alpha_1, \alpha_2, ..., \alpha_n)$ is the set $\{< \alpha_1, \alpha_2, ..., \alpha_n >: A(\alpha_1, \alpha_2, ..., \alpha_n)\}$.

Concepts on individuals

The simplest kind of concepts:
$\alpha A(\alpha)$, where α is any individual variable and $A(\alpha)$ is a formula of first-order logic without quantifiers with only one free variable α.

Formula $A(\alpha)$ expresses the *intension* of a concept $\alpha A(\alpha)$.

The *extension* of a concept $\alpha A(\alpha)$ is the set $\{\alpha : A(\alpha)\}$.

Logical intensional relations between concepts

The intension of a concept $\alpha A(\alpha)$ is *a logical part* of the intension of a concept $\alpha B(\alpha)$ **iff** $B(\alpha) \vdash A(\alpha)$.

The intension of a concept $\alpha A(\alpha)$ is *logically incompatible* with the intension of a concept $\alpha B(\alpha)$ **iff** $A(\alpha) \vdash \neg B(\alpha)$.

The intension of a concept $\alpha A(\alpha)$ is *logically 'inversely' incompatible* with the intension of a concept $\alpha B(\alpha)$ **iff** $\neg A(\alpha) \vdash B(\alpha)$.

Intensional relations between concepts in a theory

The intension of a concept $\alpha A(\alpha)$ is *a part* of the intension of a concept $\alpha B(\alpha)$ *in a theory* **T iff** $\mathbf{T}, B(\alpha) \vdash A(\alpha)$.

The intension of a concept $\alpha A(\alpha)$ is *incompatible* with the intension of a concept $\alpha B(\alpha)$ *in a theory* **T iff** $\mathbf{T}, A(\alpha) \vdash \neg B(\alpha)$.

The intension of a concept $\alpha A(\alpha)$ is *'inversely' incompatible* with the intension of a concept $\alpha B(\alpha)$ *in a theory* **T iff** $\mathbf{T}, \neg A(\alpha) \vdash B(\alpha)$.

Factual intensional relations between concepts

The intension of a concept $\alpha A(\alpha)$ is *a factual part* of the intension of a concept $\alpha B(\alpha)$ **iff** there are true sentences $C_1, C_2, ..., C_k$ such that $C_1, C_2, ..., C_k, B(\alpha) \vdash A(\alpha)$.

The intension of a concept $\alpha A(\alpha)$ is *factually incompatible* with the intension of a concept $\alpha B(\alpha)$ **iff** there are true sentences $C_1, C_2, ..., C_k$ such that $C_1, C_2, ..., C_k, A(\alpha) \vdash \neg B(\alpha)$.

The intension of a concept $\alpha A(\alpha)$ is *factually 'inversely' incompatible* with the intension of a concept $\alpha B(\alpha)$ **iff** there are true sentences $C_1, C_2, ..., C_k$ such that $C_1, C_2, ..., C_k, \neg A(\alpha) \vdash B(\alpha)$.

Two approaches to Syllogistic and to semantics of categorical statements

1. Syllogistic as a logic of concepts extensions

Extensional interpretation of categorical statements:

Every general term represents *a set of individuals*.
Syllogistic constants a, i, e, o are considered as the signs of different *relations between two sets*.

2. Syllogistic as a logic of concepts intensions

Intensional interpretation of categorical statements:

Every general term is connected with *a formula of first-order language*.
Syllogistic constants represent *logical relations between two formulas*, notably, logical *derivability*.

Extensional semantics for syllogistic

Model is a pair $\mathcal{M} = \langle \mathbf{D}, \varphi \rangle$, where

1. $\mathbf{D} \neq \varnothing$ (non-empty set of individuals);
2. φ is a function assigning a subset of $\mathbf{D}$ to each term P: $\varphi(P) \subseteq \mathbf{D}$.

Truth definitions:

$$\mathcal{M} \models SaP \text{ iff } \varphi(S) \subseteq \varphi(P),$$
$$\mathcal{M} \models SeP \text{ iff } \varphi(S) \cap \varphi(P) = \varnothing,$$
$$\mathcal{M} \models SiP \text{ iff } \varphi(S) \cap \varphi(P) \neq \varnothing,$$
$$\mathcal{M} \models SoP \text{ iff } \varphi(S) \setminus \varphi(P) \neq \varnothing,$$

usual for complex formulas.

Formula A is **valid** (in extensional semantics) iff $\mathcal{M} \models A$ in every model $\mathcal{M}$.

'Fundamental' syllogistic

Axioms:

A0. Propositional tautologies,

A1. $(MaP \wedge SaM) \supset SaP$,

A2. $(MeP \wedge SaM) \supset SeP$,

A3. $SeP \supset PeS$,

A4. SaS,

A5. $SiP \supset SiS$,

A6. $SoP \supset SiS$,

A7. $SiP \equiv \neg SeP$,

A8. $SoP \equiv \neg SaP$.

Rule: *modus ponens*.

Formula A is a theorem of **FS** iff A is valid in extensional semantics.

Łukasiewicz' syllogistic

Additional axiom:

A9. $SiP \vee SoP$.

Formula A is a theorem in **ŁS** iff A is valid in every model $\mathcal{M} = \langle \mathbf{D}, \varphi \rangle$ such that $\varphi(P) \neq \varnothing$ for each term P.

Intensional semantics for syllogistic

Let $\mathcal{F}$ be the set of all first-order formulas in the sublanguage without quantifiers, with one individual variable x and primitive connectives $\wedge$, $\vee$ and $\neg$.

δ is a function assigning a formula from $\mathcal{F}$ to each general term P: $\delta(P) \in \mathcal{F}$.

Truth definitions:

(1) $\delta \models SaP$ iff $\delta(S) \vdash \delta(P)$, i.e. the intension of P is a part of the intension of S,

(2) $\delta \models SeP$ iff $\delta(S) \vdash \neg\delta(P)$, i.e. the intension of S is incompatible with the intension of P,

(3) $\delta \models SiP$ iff $\delta(S) \nvdash \neg\delta(P)$,

(4) $\delta \models SoP$ iff $\delta(S) \nvdash \delta(P)$,

(5) usual for complex formulas.

Intensional semantics for syllogistic

Formula A is **valid** (in intensional semantics) iff $\delta \models A$ for any assignment δ.

Formula A is a theorem of **FS** iff A is valid in intensional semantics.

Formula A is a theorem of **ŁS** iff $\delta \models A$ for any assignment δ such that $\neg\delta(P)$ is not a theorem of first-order logic for each term P.

‘Relevant’ interpretation of syllogistic

The core idea:
Let us try to replace the classical derivability ‘$\vdash$’ with non-classical logical relations in the intensional interpretation of syllogistic formulas.

Let ‘$\vDash_{rel}$’ be the first-degree entailment (system **FDE**).

Truth definitions:

(1) $\delta \models SaP$ iff $\delta(S) \vDash_{rel} \delta(P)$,

(2) $\delta \models SeP$ iff $\delta(S) \vDash_{rel} \neg\delta(P)$,

(3) $\delta \models SiP$ iff $\delta(S) \nvDash_{rel} \neg\delta(P)$,

(4) $\delta \models SoP$ iff $\delta(S) \nvDash_{rel} \delta(P)$,

(5) usual for complex formulas.

A is **relevantly valid** iff $\delta \models A$ for any assignment δ.

‘Relevant’ interpretation of syllogistic

Some theorems of **FS** and **ŁS** are not relevantly valid.

Let's consider the axioms $SiP \supset SiS$ and $SoP \supset SiS$.

Let $\delta(S) = \mathrm{Q}^1(x) \wedge \neg\mathrm{Q}^1(x)$ and $\delta(P) = \mathrm{R}^1(x)$.

Since $\mathrm{Q}^1(x) \wedge \neg\mathrm{Q}^1(x) \not\models_{rel} \neg\mathrm{R}^1(x)$, so $\delta \models SiP$.

Since $\mathrm{Q}^1(x) \wedge \neg\mathrm{Q}^1(x) \not\models_{rel} \mathrm{R}^1(x)$, so $\delta \models SoP$.

But $\mathrm{Q}^1(x) \wedge \neg\mathrm{Q}^1(x) \models_{rel} \mathrm{Q}^1(x) \wedge \neg\mathrm{Q}^1(x)$, so $\delta \not\models SiS$.

Hence, $SiP \supset SiS$ (A5) and $SoP \supset SiS$ (A6) are not relevantly valid.

Syllogistic IFS

IFS is **FS** without axioms A5 and A6.

Axioms:

- A0. Propositional tautologies,
- A1. $(MaP \wedge SaM) \supset SaP$,
- A2. $(MeP \wedge SaM) \supset SeP$,
- A3. $SeP \supset PeS$,
- A4. SaS,
- A7. $SiP \equiv \neg SeP$,
- A8. $SoP \equiv \neg SaP$.

Rule: *modus ponens*.

Formula A is a theorem of **IFS** iff A is relevantly valid.

Are the extensional and intensional interpretations of categorical statements equivalent?

Let us consider the sentence of the form *SaP* '*All ruminant animals are cloven-hoofed*'.

According to extensional interpretation, this sentence is *true*, because the extension of its subject is the subset of the extension of its predicate factually.

According to intensional interpretation, this sentence is *false*, because the intension of the concept 'cloven-hoofed animals' is not a logical part of the intension of 'ruminant animals'.

The statement 'The extension of S is factually a subset of the extension of P'
is not equivalent
to the statement 'The intension of P is a logical part of the intension of S.

Are the extensional and intensional interpretations of categorical statements equivalent?

Let us consider the sentence of the form *SiP* '*Some clowns are multi-millionaires*'.

According to extensional interpretation, this sentence is *false*, because factually there is no clown who is multi-millionaire.

According to intensional interpretation, this sentence is *true*, because the intension of the concept 'clown' is logically compatible with the intension of 'multi-millionaire'.

The statement 'The intersection of the extensions of S and P is not empty'
is not equivalent
to the statement 'The intension of S is logically compatible with the intension of P'.

Syllogistic of factual extensions and logical intensions

We use *standard* syllogistic constants *a*, *i*, *e*, *o* for categorical statements under their *extensional* interpretation.

We use *modal* syllogistic constants $a^{\Box}$, $e^{\Box}$, $i^{\Diamond}$, $o^{\Diamond}$ for the statements under their *intensional* interpretation.

Let $\mathcal{F}$ be the set of all first-order formulas in the sublanguage without quantifiers, with one individual variable x and primitive connectives $\wedge$, $\vee$ and $\neg$.

Model is a triple $\mathcal{M} = \langle \delta, \mathbf{D}, \varphi \rangle$, where

1. $\delta(P) \in \mathcal{F}$ for each general term P;
2. $\mathbf{D} \neq \varnothing$ (non-empty set of individuals);
3. φ is a function assigning a subset of $\mathbf{D}$ to each predicate symbol Q^1: $\varphi(\mathrm{Q}^1) \subseteq \mathbf{D}$.

Factual extensions of general terms

Let Φ be the function assigning a subset of $\mathbf{D}$ to every formula from $\mathcal{F}$:

1. $\Phi(\mathrm{Q}^1(x)) = \varphi(\mathrm{Q}^1)$;
2. $\Phi(\neg A) = \mathbf{D} \backslash \Phi(A)$;
3. $\Phi(A \wedge B) = \Phi(A) \cap \Phi(B)$;
4. $\Phi(A \vee B) = \Phi(A) \cup \Phi(B)$.

$\Phi(\delta(S))$ is the *factual extension* of general term S in the model $\mathcal{M}$.

Truth definitions

Truth definitions of modal and standard syllogistic formulas in a model $\mathcal{M}$:

1. $\mathcal{M} \models Sa^{\Box}P$ iff $\delta(S) \vdash \delta(P)$;
2. $\mathcal{M} \models Se^{\Box}P$ iff $\delta(S) \vdash \neg\delta(P)$;
3. $\mathcal{M} \models Si^{\Diamond}P$ iff $\delta(S) \nvdash \neg\delta(P)$;
4. $\mathcal{M} \models So^{\Diamond}P$ iff $\delta(S) \nvdash \delta(P)$;
5. $\mathcal{M} \models SaP$ iff $\Phi(\delta(S)) \subseteq \Phi(\delta(P))$;
6. $\mathcal{M} \models SeP$ iff $\Phi(\delta(S)) \cap \Phi(\delta(P)) = \varnothing$;
7. $\mathcal{M} \models SiP$ iff $\Phi(\delta(S)) \cap \Phi(\delta(P)) \neq \varnothing$;
8. $\mathcal{M} \models SoP$ iff $\Phi(\delta(S)) \backslash \Phi(\delta(P)) \neq \varnothing$;
9. usual for complex formulas.

A syllogistic formula A is *valid* iff $\mathcal{M} \models A$ in any model $\mathcal{M}$.

Thank you!