Skolem's paradox as logic of ground

The great Norwegian scientist, mathematician and logician Thoralf Skolem (1922) introduced the "relativity of the notion of set" and the "improper interpretation" of a set-theory structure, based on the axiom of choice. One can demonstrate that they can serve as a ground of the logic of ground.

The thesis is: The "improper interpretation" of an infinite set-theory structure founds the "proper interpretation" and thus that structure self-founds itself as the one interpretation of it can found the other.

Furthermore, the interesting corollaries are implied about the theory of information for information can be introduced as a mapping of the proper into the improper interpretation of a mathematical structure being also a quantity of its complexity in the sense of Kolmogorov, Chaitin, and Martin-Löf (1965-1977). Thus involved, the quantity of information can be interpreted as the quantity of "substance" or "being" of that structure allowing of self-grounding.

The innovation cited in the beginning and known also as "Skolem's paradox" can be discussed as a special property of infinity implying the concept of choice and thus of information:

Indeed any finite set can be well-ordered without utilizing the axiom of choice. However that well-ordering is trivial and coincides with the identical mapping of the set into itself. In other words, the "improper interpretation" of any finite set is identical to the "proper one" of it.

However the same statement can be deduced in no way as to any infinite set and should be postulated being known as the well-ordering theorem which is equivalent to the axiom of choice.

Consequently, the sense of "Skolem's paradox" is to demonstrate that the axiom of choice is equivalent to treating any infinite set not only as a countable one, but also even as a finite one just involving the "relativity of the notion of set". Even more, any finite set can be in turn interpreted as an infinite one once the axiom of choice is admitted.

Consequently, the extreme sense of "Skolem's paradox" should be that the axiom of choice is equivalent to a one-to-one mapping of any finite into any infinite set therefore involving the concept of choice and thus of information as follows: That mapping required by the axiom of choice and requiring it in turn should be grounded on a generalized function (Schwartz distribution) replacing it with an usual function randomly choosing only one value of the function for any value of the independent variable (argument). Consequently, the concept of choice can be defined by a pair of a generalized function and a usual one obeying the above condition. Furthermore, the quantity of information is definable as some mapping of those pairs into a number set. This is a generalized introduction of the quantity of information reducible to the standard definition of it where the set of the function values is finite for any value of the independent variable. It includes the quantity of quantum information in the case where that set is infinite for at least one value of the argument.

The Kolmogorov – Chaitin – Martin-Löf measure of information as complexity can be obtained by the well-ordering of that mapping choice by choice.

The special Skolem property of infinity implying both concepts of choice and information can be summarized so: If the axiom of choice is given, any infinite set possesses always an improper finite interpretation, which implies in turn that any finite set possesses an improper infinite interpretation. Thus the introduction of infinity under the axiom of choice implies the doubling of any mathematical structure either finite or infinite with an improper twin interpretation correspondingly either infinite or finite.

That property can ground any consistent mathematical structure in itself by itself so: The improper interpretation can be always interpreted as the ground of the proper one and the improper and proper interpretations are isomorphic to each other. Consequently, that structure self-founds by the mediation of the improper twin interpretation. Thus the introduction of infinity in the pair of infinity and finiteness implies the self-foundation of any consistent mathematical structure under the condition of the axiom of choice. In other words, the concepts of choice and information by means of the axiom of choice implies for infinity and finiteness to be twined as two copies or two interpretations of some joint structure. However if the axiom of choice does not hold, a gap divides finiteness of infinity. The same can be literally repeated as to the grounding and grounded, too: Thus the axiom of choice twins also them implying self-foundation. Its rejection divides them correspondingly into a meta-theory and a theory incommensurate to each other.

The so-called completeness theorem of Gödel (1930) can serve as an illustration: The finite set of axioms and the infinite set of theorems implied by the formers under the conditions of the cited theorem can be considered as the above twin interpretations implying self-foundation (completeness). Accordingly the so-called first incompleteness theorem of Gödel (1931) can be interpreted in the same context as the impossibility of an infinite set of statements to ground itself onto another infinite set of axioms (as including Peano's axioms): Only the pair of infinity and finiteness can self-ground rather than that of two infinities.

Zermelo's strategy (1908) of the foundation of set theory in a consistent way is another illustration: A set can be accepted to be a set if and only if it is a true subset of another set. This excludes the universal set. However its sense in the above context is different: One can choose a relevant subset (B) of the complement to any set (A) so that the pair of A and B to be considered correspondingly as one of the proper and improper interpretation of some joint structure. Even more, Skolem's innovation can serve as the base of a rigorous proof for consistency of the ZFC set theory.

In fact, Skolem's "paradox" is a breakthrough into the true understanding of infinity and its ability to constitute a self-grounding pair with finiteness.